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Laplace's method on a computer algebra system with an application to the real valued modified Bessel functions

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Abstract

We examine a *Maple* implementation of two distinct approaches to Laplace's method used to obtain asymptotic expansions of Laplace-type integrals. One algorithm uses power series reversion, whereas the other expands all quantities in Taylor or Puiseux series. These algorithms are used to derive asymptotic expansions for the real valued modified Bessel functions of pure imaginary order and real argument that mimic the well-known corresponding expansions for the unmodified Bessel functions. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Two of the most important tools in an applied mathematician's toolbox are Watson's lemma and Laplace's method. The first determines the asymptotic nature of integrals of the form

$$h(v) = \int_0^\infty q(t)\mathrm{e}^{-vt}\,\mathrm{d}t \tag{1.1}$$

in the limit as $v \to +\infty$, where $q(t)$ is a real or complex valued function. The function $h(v)$ is known as the *Laplace transform* of $q(t)$. Laplace's method is used on a more general form of (1.1), namely

$$H(v) = \int_a^b q(t)\mathrm{e}^{-vp(t)}\,\mathrm{d}t, \tag{1.2}$$

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where $q(t)$ and $p(t)$ are real or complex valued functions and a, b are either finite real numbers, positive infinity or negative infinity. Throughout this work, we assume that v and $p(t)$ are real and that $p(t)$ is positive on $[a, b]$. Integrals of the form (1.2) are referred to as *Laplace-type integrals* because of their similarity to (1.1).

There are two basic approaches to Laplace's method found in standard textbooks that discuss the topic. The first approach, as discussed for example in [24], is mathematically rigorous and requires the reversion of a power series expansion. The second approach, as discussed for example in [3], is more of a brute force approach: it directly attacks the problem, carrying a large number of terms in series expansions during computations (often too many terms) and neglects nonrelevant terms in the end. Both methods yield the same expansions, and both are algebraically intensive. Erdélyi [15] states that these two methods are *computationally* equivalent. This paper examines the implementation of these approaches in *Maple* [10], where algebraic manipulations are carried out quickly and accurately. The choice of *Maple* as opposed to any other computer algebra system is purely one of convenience to us. Let us be clear that we are not comparing the different approaches per se, but rather their implementation on a computer algebra system. There is no question that the reversion approach is mathematically rigorous and in principle is the correct method to use.

We describe our algorithm for the reversion approach in Section 2 and brute force approach in Section 3. A comparison of the two algorithms is made in Section 4. An implementation of these codes concludes the paper in Section 5, where we obtain Debye asymptotic expansions for the real valued modified Bessel functions and their derivatives for purely imaginary order and real arguments.

Terminology. Throughout this work, we denote a Puiseux series expansion of a function $f(z)$ in the form

$$f(z) = f_0 + f_1(z-z_0)^{\gamma_1} + f_2(z-z_0)^{\gamma_2} + f_3(z-z_0)^{\gamma_3} + \cdots, \tag{1.3}$$

where $f_1 \neq 0$ and the exponents γ_s, $s = 1, 2, 3, \ldots$, form a set of positive increasing numbers. We symbolically represent the right-hand side of (1.3) by $\mathscr{T}\{f(z), z_0\}$. We say that an expansion of the form (1.3) truncated to m terms has precision P if

$$f(z) = f_0 + \sum_{s=1}^{m} f_s(z-z_0)^{\gamma_s} + \mathrm{O}((z-z_0)^P),$$

where $f_0, f_1, \ldots, f_m$ are all known (where all but f_1 may vanish) and the notation $\mathrm{O}(\cdot)$ represents the order of the first neglected term, i.e., $\gamma_m < P$. If $P < 0$, then the expansion in (1.3) is *asymptotic*, the exponents γ_s form a set of negative decreasing numbers, and $|\gamma_m| < |P|$. The actual value of m is irrelevant since *Maple* automatically determines m given a precision P.

2. Reversion approach

2.1. Reversion of Puiseux series

The bulk of the work in the reversion approach to Laplace's method will lie in the reversion of a Puiseux series as given in (1.3). We, therefore, begin with this topic. Proofs of Theorems 2.1 and 2.2 can be found in standard texts in complex analysis, e.g., [13]. The first is the strong form of the Implicit Function Theorem.

Theorem 2.1. *Let $f(z)$ be analytic*[2] *at $z = z_0$, $f(z_0) = w_0$, and $f'(z_0) \neq 0$. Then the equation $w = f(z)$ has a unique solution $z = F(w)$ such that $F(w_0) = z_0$ and $F(w)$ is analytic at w_0.*

It follows that the functions $f(z)$ and $F(w)$ have the expansions

$$f(z) = f_0 + f_1(z - z_0) + f_2(z - z_0)^2 + f_3(z - z_0)^3 + \cdots, \tag{2.1}$$

$$F(w) = F_0 + F_1(w - w_0) + F_2(w - w_0)^2 + F_3(w - w_0)^3 + \cdots, \tag{2.2}$$

where $f_1 \neq 0$. Clearly, $F_0 = z_0$ and $w_0 = f_0$. The remaining coefficients F_s can be expressed in terms of f_s by directly inserting (2.1) into (2.2) and equating like powers of $z - z_0$. From this it follows that if $f(z)$ has precision N, then the precision of the reverted series is at best N. The exception is when $f(z)$ is a polynomial, in which case the precision of $F(w)$ is unbounded. Lagrange [20] cleverly manipulated power series to obtain an algorithm for computing the coefficients F_s:

Theorem 2.2 (Lagrange reversion formula). *For $s = 1, 2, \ldots,$*

$$F_s = \frac{1}{s!}\left[\frac{\mathrm{d}^{s-1}}{\mathrm{d}z^{s-1}}\left\{\frac{z - z_0}{f(z) - f_0}\right\}^s\right]_{z=z_0}.$$

Though Theorem 2.2 is powerful, it is cumbersome to use directly. A simpler implementation of Theorem 2.2 is given by the following pseudo-code.

```
F0 = z0
t1 = (z − z0)/(f(z) − f0)
F1 = coeff(t1, z − z0, 0)
t2 = t1
for s = 2, N do
  t2 = t2 ∗ t1
  Fs = coeff(t2, z − z0, s − 1)/s
end do
```

Tabulation of the first several coefficients F_j can be found in [12,17,22,35]. Theoretical and algorithmic development for the reversion of systems of equations were explored in [6,11,19,25,26,34,37] and finite precision codes have been developed in [7–9,32,33]. Algorithms for reversion for use on computer algebra systems that use generating functions appeared in [4,29]. The reversion of asymptotic expansions appeared in [5,16] for expansions in inverse powers of the argument and in [27,28] for nested expansions.

Many series expansions which fail to meet the requirements of Theorems 2.1 and 2.2 can be manipulated to use them, though uniqueness of the reverted solution is lost. We classify the expansion in (1.3) into four types of systematic patterns, where $f(z)$ and $F(w)$ are as follows:

Type I: Eqs. (2.1)–(2.2).

[2] By "$f(z)$ is analytic at $z = z_0$" we mean that $f(z)$ is differentiable in a neighborhood of z_0.

Type II: Those given by the equations

$$f(z) = f_0 + \sum_{s=1}^{\infty} f_s(z - z_0)^{s-1+\beta}, \quad F(w) = \sum_{s=0}^{\infty} F_s(w - w_0)^{s/\beta}, \tag{2.3}$$

where $\beta > 0$.

Type III: Those given by the equations

$$f(z) = \sum_{s=0}^{\infty} f_s(z - z_0)^{s\beta}, \quad F(w) = F_0 + \sum_{s=1}^{\infty} F_s(w - w_0)^{1/\beta+s-1}, \tag{2.4}$$

where $\beta > 0$.

Type IV: Those given by the equations

$$f(z) = f_0 + \sum_{s=1}^{\infty} f_s(z - z_0)^{(\alpha-\beta+s-1)/\beta},$$

$$F(w) = F_0 + \sum_{s=1}^{\infty} F_s(w - w_0)^{(s-1+\beta)/(\alpha-\beta)}, \tag{2.5}$$

where $\alpha, \beta > 0$.

In all cases, $F(w_0)=z_0$ and $w_0 = f_0$. Note that this classification is not unique: Type III is a subset of Type II if β in (2.4) is an integer and vice versa if $1/\beta$ in (2.3) is an integer. A large majority of the functions typically encountered when analyzing (1.2) fall into one of these categories.

Given a generic series expansion as returned by *Maple* of the form (1.3), the first task is to determine the expansion's type. This is accomplished by examining the exponents γ_1, γ_2, and γ_3.

Types I–III: We determine that (1.3) is one of Types I–III as follows:

I $\gamma_1 = 1$ and γ_2, γ_3 are integers;
II $\gamma_2 - \gamma_1$, $\gamma_3 - \gamma_1$, and $\gamma_3 - \gamma_2$ are all integers;
III γ_2/γ_1 and γ_3/γ_1 are integers.

We then perform the transformations

I $\xi = z - z_0$, $v = w - w_0$, $l(\xi) = f(z) - f_0$,
II $\xi = z - z_0$, $v = (w - w_0)^{1/\beta}$, $l(\xi) = (f(z) - f_0)^{1/\beta}$,
III $\xi = (z - z_0)^{\beta}$, $v = w - w_0$, $l(\xi) = f(z) - f_0$.

For these types, the resulting equation to be reverted is $v = l(\xi)$, where

$$l(\xi) = l_1\xi + l_2\xi^2 + l_3\xi^3 + \cdots.$$

We use Theorem 2.2 to obtain the reverted expansion $\xi = L(v)$, where

$$L(v) = L_1 v + L_2 v^2 + L_3 v^3 + \cdots.$$

Finally, we transform back to original variables to obtain the reverted expansion.

Type IV: If an expansion of the form (1.3) fails to be classified into one of Types I–III, we attempt to classify as Type IV as follows. We define β as the least common multiple of the denominators

Table 1
The precision of various expansions as we proceed through the transformations described in the text. The numbers have been chosen so that the precision of $F(w)$ is as close to N as possible

Type	Functions			
	$f(z)$	$l(\xi)$	$L(v)$	$F(w)$
I	N	N	N	N
II	$(N+1)\beta-1$	$N\beta$	$N\beta$	N
III	$(N+1)\beta$	$N+1$	$N+1$	$N+1/\beta$
IV	$((N(\alpha-\beta)+2-\beta)$ $\times(\alpha-\beta)-1)/\beta$	$(N(\alpha-\beta)+2-\beta)$ $\times(\alpha-\beta)-1$	$N(\alpha-\beta)+2-\beta$	$N+(2-\beta)/(\alpha-\beta)$

of $\gamma_1, \gamma_2, \gamma_3$. α then follows by adding β to the numerator of γ_1. We test our obtained values by verifying that $\beta-\alpha$ added to the numerators of γ_2 and γ_3 yield integers and that $\gamma_3\beta-\gamma_2\beta \geqslant 0$. Should a satisfactory solution pair $\{\alpha,\beta\}$ be found, then we perform the transformation

$$\xi=(z-z_0)^{1/\beta}, \quad v=w-w_0, \quad l(\xi)=f(z)-f_0.$$

We then recursively call the reversion procedure since $l(\xi)$ is now Type II. We obtain the reverted solution $\xi=L(v)$ and transform back to original variables.

A key element in computing the reverted series $F(w)$ is that the expansion of $f(z)$ contains a sufficient number of terms. Table 1 lists the precisions of the main and intermediate functions for each of the types. It is, of course, prudent to initially carry a few more terms.

Maple has an intrinsic command called reversion in its powseries package which operates on formal power series as defined by *Maple*. It will be shown in Section 4 that this command is faster than direct implementation of Theorem 2.2 as described above. The command solve(f(x) = y, x) also can return a series solution provided $f(x)$ initially is a series. The difficulty with this command is that it will return more solutions than we need, which in turn uses valuable computer resources and increases computation time. When the solve command returns a power series, care must be taken in choosing the desired solution, i.e., the one that satisfies $F(w_0)=z_0$. For example, the solutions obtained by using the solve command on the equation $w=z-z^2$ have the power series expansions $z=1-w-w^2+\cdots$ and $z=w+w^2+\cdots$. Clearly, the second solution is the one desired, since in our notation, $z_0=w_0=0$. Because of these and similar complications, we avoid using the solve command.

Finally, it is crucial in our work to determine the exponents γ_1, γ_2, and γ_3. If the series is a standard Taylor series, then γ_1 can be determined from *Maple*'s data structure of the series using the command op(2,series(f − f0, z = z0)). The data structure is not as well defined when the expansion for $f(z)$ is a Puiseux series. In this case, we attempt to determine γ_1 by computing the series expansion of $\ln(f(z))$, converting it to a polynomial in $\ln(z)$ or $\ln(1/z)$, and obtain γ_1 or $-\gamma_1$, respectively, by extracting the polynomial's leading term's coefficient. If these three methods fail, the *Maple* code returns an error. Once we know γ_1, then γ_2 trivially follows using the same procedure with $f-f_0$ replaced by $f-f_0-f_1(z-z_0)^{\gamma_1}$. Similarly for γ_3. To speed up computations, our codes allow for an optional input list containing the exponents $[\gamma_1,\gamma_2,\gamma_3]$.

2.2. Laplace's method

We are now in a position to state the main theorems, both of which we quote from [24] where the reader is directed for detailed proofs. Furthermore, references for Laplace's method and its extensions can be found in [15, Chap. 2.4]. We begin with Watson's lemma [36] which is used on integrals of the form (1.1):

Theorem 2.3 (Watson's lemma). *Let $q(t)$ be a function of the positive real variable t, such that*

$$q(t) \sim \sum_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu} \quad (t \to 0), \tag{2.6}$$

where λ and μ are positive constants, and let v be a real variable. Then

$$\int_0^{\infty} q(t)\mathrm{e}^{-vt}\,\mathrm{d}t \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right)\frac{a_s}{v^{(s+\lambda)/\mu}} \quad (v \to +\infty), \tag{2.7}$$

provided that this integral converges throughout its range for all sufficiently large v.

It is instructive to sketch a proof for the simplest case in which $q(t)$ is infinitely differentiable in $[0,\infty)$. Suppose that $\lambda=\mu=1$, asymptoticity in (2.6) is replaced by equality, and the right-hand side of (2.6) has an infinite radius of convergence. Repeated integration by parts of the left-hand side of (2.7) n times produces

$$h(v) = \sum_{s=0}^{n-1} \frac{q^{(s)}(0)}{v^{s+1}} + \frac{1}{v^n}\int_0^{\infty} \mathrm{e}^{-vt} q^{(n)}(t)\,\mathrm{d}t,$$

where n is an arbitrary nonnegative integer. If $q^{(s)}(t) = \mathrm{O}(\mathrm{e}^{\sigma t})$ for $t \in [0,\infty)$, $s = 1,2,3,\ldots,$ then the right-hand-most term in the above equation is $\mathrm{O}(v^{-n}(v-\sigma)^{-1})$, and we conclude that

$$h(v) \sim \sum_{s=0}^{\infty} \frac{q^{(s)}(0)}{v^{s+1}} \quad (v \to +\infty).$$

Asymptoticity in (2.7) is justified by a more detailed examination of the remainder term.

Integrals of the form (2.7) are analyzed using

Theorem 2.4 (Laplace's method). *Suppose*

(1) *$p(t)$ is a real function independent of v, and let $q(t)$ be a real or complex function.*
(2) *$p(t) > p(a)$ when $t \in (a,b)$, and for every $c \in (a,b)$, the infimum of $p(t)-p(a)$* in *$[c,b)$ is positive.*
(3) *$p'(t)$ and $q(t)$ are continuous in a neighborhood of a, except possibly at a.*
(4) *As $t \to a^+$,*

$$p(t) \sim p(a) + \sum_{s=0}^{\infty} p_s(t-a)^{s+\mu} \tag{2.8}$$

and

$$q(t) \sim \sum_{s=0}^{\infty} q_s (t-a)^{s+\lambda-1}, \tag{2.9}$$

where the first of these relations is differentiable, μ is a positive constant and λ is a real or complex constant such that $\Re(\lambda) > 0$.

Then

$$H(v) \sim \mathrm{e}^{-vp(a)} \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{v^{(s+\lambda)/\mu}} \quad (v \to +\infty), \tag{2.10}$$

where the coefficients a_s are those found in the series expansion $g(\tau) = q(t)/p'(t)$ about $t = a$, where

$$t - a \sim \sum_{s=1}^{\infty} c_s \tau^{s/\mu} \quad (v \to 0^+) \tag{2.11}$$

obtained by reverting the change of variables $\tau = p(t) - p(a)$.

The first point to make is that though $p(t)$ in (2.8) and $q(t)$ in (2.9) are Type II, the theorem is valid should these functions be Type III or IV, though the form of the expansion in (2.10) will be different.

Laplace's method, as stated here, transforms the integral (1.2) into the form

$$H(v) = \int_0^{p(b)-p(a)} g(\tau)\mathrm{e}^{-v\tau}\,\mathrm{d}\tau. \tag{2.12}$$

This equation is symbolically exact. However, in most situations, $g(\tau)$ is not known analytically. Rather, we know only its series expansion about $\tau = 0$. Upon substituting this expansion into (2.12), the upper limit can be replaced by infinity by arguing that the integrand's contribution to

$$H(v) \doteqdot \int_0^{\infty} g(\tau)\mathrm{e}^{-v\tau}\,\mathrm{d}\tau \tag{2.13}$$

on the interval $(p(b) - p(a), \infty)$ is exponentially small. Note that this may not hold for $g(\tau)$ formally since the function may not be defined on the entire interval or may have singularities, poles, etc. (A rigorous proof can be found in [24].) At this stage, one can use Watson's lemma with $g(\tau)$ playing the role of $q(t)$ in Theorem 2.3. The power of this transformation is that the remainder term associated with truncating the sum in (2.10) to a finite number of terms has already been examined in Watson's lemma. Thus, asymptoticity in (2.10) is guaranteed.

The algorithm for the *Maple* code Laplace_rev which implements the reversion approach to Laplace's method follows the sketch of the proof to Theorem 2.4. The routine accounts for $p(t)$ and $q(t)$ being any of Types I–IV, and accepts as input the functions $p(t), q(t)$, the variables v, t and a, and the number N, where $-N$ is the desired precision of the returned expansion, where the variables are as in (1.2), and where the minimum of $p(t)$ on the interval $[a, b]$ occurs only at $t = a$. We first compute the reversion of $\tau = p(t) - p(a)$ to obtain $t - a$ as a Puiseux expansion in τ (i.e. (2.11) if $p(t)$ is Type II), then compute a power series expansion for $g(\tau) = q(t)/p'(t)$

(i.e., $g(\tau)=\sum_{s=0}^{\infty} a_s\tau^{(\lambda+s-\mu)/\mu}$ if $q(t)$ and $p(t)$ are both Type II), and finally symbolically evaluate (2.13) (to obtain (2.10) if $q(t)$ and $p(t)$ are both Type II). For the four types of Puiseux series considered in the previous section, the reverted series for $t-a$ will be a series in τ in powers of $1/\mathsf{numer}(\gamma_1)$, where $\mathsf{numer}\,()$ denotes the numerator of a rational expression. Thus, if we desire that the final expansion for $H(v)$ to have precision $-N$ in v, then all series expansions in the routine Laplace_rev must have precision at least $N\times\mathsf{numer}(\gamma_1)$ in τ. In implementing this approach in *Maple*, we encountered two problems in computing (2.13). The first is that $g(\tau)$ must be converted to an expansion without an explicit order because *Maple* cannot compute the expression $\int \mathrm{O}(\tau^P)\,\mathrm{d}\tau$. The second problem is that *Maple* appears to express $g(\tau)$ as a single fraction with a common denominator before integrating. This process is extremely time consuming and can be circumvented when $g(\tau)$ is a traditional Taylor series. In this special case, we compute the integral term by term in a do-loop since the coefficients and exponents are easily determined from the series' data structure.

3. Brute force method

The brute force method follows a heuristic approach. We assume that the conditions of the final paragraph of Section 2 still hold. Then, the main contribution to the integral (1.2) in the limit as $v\to+\infty$ is localized about the point where the exponent function $p(t)$ attains an absolute minimum on $[a,b]$,[3] that is, at $t=a$. Thus, (1.2) is approximated by

$$H(v) \risingdotseq \int_a^{a+\delta} q(t)\mathrm{e}^{-vp(t)}\,\mathrm{d}t, \tag{3.1}$$

where δ is a small arbitrary parameter such that the main contribution of the integrand in (1.2) to $H(v)$ occurs on the interval $[a,a+\delta]$. The functions $p(t)$ and $q(t)$ in (3.1) are now replaced by their series expansions about $t=a$, and the integral takes the form

$$H(v) \risingdotseq \mathrm{e}^{-vp(a)}\int_a^{a+\delta} \mathscr{T}\{q(t),a\}\mathrm{e}^{-v(t-a)^{\gamma_1}}\chi\{-vp(t),a\}\,\mathrm{d}t,$$

where $\mathscr{T}\{f(z),z_0\}$ was defined at the end of Section 1 and

$$\chi\{f(t),a\}=\mathscr{T}\{\exp[f(t)-f(a)-f_1(t-a)^{\gamma_1}],a\}.$$

The next step is to replace the upper limit $a+\delta$ by ∞, a process justified by arguing that the integrand's contribution on the interval $(a+\delta,\infty)$ is exponentially small as before. Again, this process may not be valid if we were not considering series expansions. The resulting integral is

$$H(v) \risingdotseq \frac{\mathrm{e}^{-vp(a)}}{v^{1/\gamma_1}}\int_0^{\infty} g(\tau,v)\mathrm{e}^{-p_1\tau^{\gamma_1}}\,\mathrm{d}\tau, \tag{3.2}$$

[3] A function $f(x)$ has an *absolute minimum* at $x=x_0$ on an interval $[x_1,x_2]$ if $f(x_0)\leqslant f(x)$ for all $x\in[x_1,x_2]$.

where $g(\tau, v) = \chi\{-vp(t), a\} \times \mathscr{T}\{q(t), a\}$ is a series expansion under the change of variables $\tau = v^{1/\gamma_1}(t-a)$. One more transformation $\tilde{\tau} = p_1\tau^{\gamma_1}$ transforms (3.2) into

$$H(v) \doteqdot \frac{e^{-vp(a)}}{\gamma_1(p_1 v)^{1/\gamma_1}} \int_0^\infty g((\tilde{\tau}/p_1)^{1/\gamma_1}, v)\tilde{\tau}^{1/\gamma_1 - 1}e^{-\tilde{\tau}}\, d\tilde{\tau}, \tag{3.3}$$

which is in the correct form for use of Watson's lemma. The astute reader will note that the series manipulations and transformations which yield the integral in (3.3) are *computationally* equivalent to the reversion of the change of variables $\tilde{\tau} = v(p(t) - p(a))$ and substitution. It is not clear, however, if the expansion for $H(v)$ obtained by computing the integrals in (3.2) or (3.3) is *asymptotic*. This is a major fault of the brute force approach. Since the expansions obtained by computing the integrals in (3.2) and (3.3) are the same as that obtained by Theorem 2.4, the assertion is validated. What the brute force approach lacks in grace it makes up for in directness of implementation.

As in the reversion approach, one of this approach's key difficulties lies in carrying enough terms (in powers of τ) in each of the series expansions so that the product function $g(\tau, v)$ in (3.2) has not neglected relevant terms in powers of v^{-1/γ_1}. *It is not important to have all the coefficients of a given power of* τ, *but rather only those coefficients that contain the relevant powers of* v^{-1/γ_1}. Suppose the expansions $\mathscr{T}\{q(t), a\}$ and $\chi\{-vp(t), a\}$ both have precision r, and let $\mu = v^{-1/\gamma_1}$ and $\mu\tau = (t-a)$. Then (3.2) has the form

$$H(\mu^{-\gamma_1}) \doteqdot \mu e^{-p(a)\mu^{-\gamma_1}} \int_0^\infty \left[\sum_{s=0}^{m_1} q_s(\tau\mu)^{\tilde{\gamma}_s} + O(\tau^r\mu^r)\right] \psi(\tau, \mu)e^{-p_1\tau^{\gamma_1}}\, d\tau, \tag{3.4}$$

where

$$\psi(\tau, \mu) = \exp\left(\mu^{-\gamma_1}\sum_{s=2}^{m_2} p_s(\tau\mu)^{\gamma_s} + O(\tau^r\mu^{r-\gamma_1})\right). \tag{3.5}$$

Again, the actual values of m_1 and m_2 are irrelevant. If we were to expand $\psi(\tau, \mu)$ in a series of τ, then

$$\psi(\tau, \mu) = \sum_{s=0}^{m_3} \tilde{\psi}_s(\mu)\tau^{\tilde{\gamma}_s} + O(\tau^r), \tag{3.6}$$

where the coefficients $\tilde{\psi}_s(\mu)$ contain powers of μ, many of which must be neglected. Instead, we expand $\psi(\tau, \mu)$ in a series of μ to obtain

$$\psi(\tau, \mu) = \sum_{s=0}^{m_4} \hat{\psi}_s(\tau)\mu^{\tilde{\gamma}_s - \gamma_1} + O(\mu^{r-\gamma_1}), \tag{3.7}$$

where the coefficients $\hat{\psi}_s(\tau)$ will contain only the relevant powers of τ and not all of them. We multiply (3.7) by the series in the square brackets of (3.4) to obtain $g(\tau, \mu^{-\gamma_1})$ with precision $(r - \gamma_1)$ in μ, or $(1 - r/\gamma_1)$ in v. Thus, if the desired precision of $H(v)$ is $-N$, then the required precision in μ is $N\gamma_1$. Consequently, $r = (N+1)\gamma_1$.

It is prudent to reorder $g(\tau, \mu^{-\gamma_1})$ and collect on like powers of τ to minimize the amount of work done in the integration step. Symbolic computation of the integrals in both (3.2) and (3.3) suffer the same fate as that of (2.13). We find that it is faster to compute (3.2) rather than (3.3) since the coefficient of $\exp(-p_1\tau^{-\gamma_1})$ in the integral of (3.2) is more likely to be a series with integer exponents than the corresponding function in (3.3). The exponent γ_1 is determined as discussed at the end of Section 2.1. The routine Laplace_brute which computes this approach to Laplace's method accepts γ_1 as an optional input variable.

4. Discussion

The goal of this paper is to examine the implementation of the algebraically intensive Laplace's method on a computer algebra system, namely *Maple*. The number of terms we are able to compute using our *Maple* codes is limited by the computing resources available. In order to see differences between the two approaches examined here, we compute asymptotic expansions to an obscenely large number of terms in our test cases. Our only measurable quantities are the computation time and memory utilization as returned by *Maple*. Differences between the codes is due, in part, to the speed with which *Maple* performs series manipulations. These manipulations can be memory intensive, and computing resources can be quickly overwhelmed.

We performed a comparison of Puiseux series reversion by direct implementation of Theorem 2.2 rev1 and by using *Maple's* reversion command rev2 for a variety of functions. There seems to be little difference in execution time between the two for low precisions and drastic differences for high precisions. Table 2 lists the computation times and resource utilization for the reversion of $\psi(\tau;\theta)$ in (5.5) of the next section to various precisions. From this table, we readily see that system resources are quickly overwhelmed using rev1. This is due to the fact that we are continuously computing integer powers of series expansions. The advantage that rev2 has is that *Maple* (presumably) takes advantage of $l(\xi)$'s data structure when converted to a formal power series as defined by *Maple*.

The two approaches to Laplace's method are tested against each other in a similar manner. Table 3 lists the same quantities for the computation of the asymptotic expansion of the modified Bessel function $K'_{i\nu}(\nu)$ to large precisions. The mathematical details of this computation again are discussed in the next section. We see in this example that the brute force method outperforms the reversion approach. We find that this holds true in a vast majority of the cases tested and therefore recommend the brute force method for use on a computer algebra system.

Table 2
Memory utilization (in megabytes rounded to the nearest whole number) and CPU time (in seconds) as returned by *Maple*, Release 5.1, for the computation of the reversion of $\psi(\tau;\theta)$ as defined in (5.5) for various precisions N. These calculations were carried out on a 500 MHz *Compaq* Alpha workstation. The procedure rev1 directly implements Theorem 2.2 and rev2 uses *Maple's* intrinsic reversion command. Both the CPU time and the memory utilization seem to be polynomial in N

N	rev1		rev2	
	CPU time	Memory utilization	CPU time	Memory utilization
10	0.411	8	0.494	8
25	12.141	136	7.658	104
50	340.169	1811	155.308	1114
75	3308.001	8678	1318.459	4821
100	19,957.122	27,671	7884.301	14,462

Table 3
Memory utilization (in megabytes rounded to the nearest whole number) and CPU time (in seconds) as returned by *Maple*, Release 5.1, for the computation of the asymptotic expansion of $K'_{\mathrm{i}\nu}(\nu)$ as discussed in Section 5.3 for various precisions N. These calculations were carried out on a 500 MHz *Compaq* Alpha workstation. The procedure Laplace_brute is the brute force approach and Laplace_rev is the reversion approach. The memory utilization is polynomial in N with the same exponent, though the CPU time appears to be exponential in N

N	Laplace_brute		Laplace_rev	
	CPU time	Memory utilization	CPU time	Memory utilization
10	0.864	12	1.982	26
25	7.956	81	17.605	211
50	163.017	740	330.472	2159
75	1619.202	3187	2855.949	10,032
100	13,366.792	9494	16,367.262	31,636

Every effort has been made to ensure that the expansions returned by the reversion and the Laplace's method routines have the correct and desired order. We have encountered a few cases where the returned expansion was a degree or two smaller than the desired order.

5. Application to the real valued modified Bessel functions

We illustrate our codes' abilities and limitations in this section by deriving Debye-type asymptotic expansions for the real valued modified Bessel functions. Many, though not all, of the expansions we obtain here were derived in [14] using a differential equation approach. The present exposition uses the *Method of Steepest Descents* on the integral representation of the functions to obtain the desired asymptotic expansions. The differential equation approach is often more desirable since that method provides explicit errors bounds for truncated expansions and closed form formulas for the polynomials used in the expansions. This point will be emphasized below.

The present investigation focuses on the real valued solutions of the modified Bessel equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}-(x^2-\nu^2)y=0, \tag{5.1}$$

where x and ν are real variables. Though any linearly independent pair of Bessel and/or Hankel functions form a fundamental solution pair to (5.1), the two traditional solutions are the *modified Bessel functions*

$$K_{\mathrm{i}\nu}(x)=\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{e}^{-\phi(t,x,\nu)}\,\mathrm{d}t,\quad I_{\mathrm{i}\nu}(x)=\frac{1}{2\pi\mathrm{i}}\int_{\infty-\mathrm{i}\pi}^{\infty+\mathrm{i}\pi}\mathrm{e}^{\phi(t,x,\nu)}\,\mathrm{d}t, \tag{5.2}$$

where $\phi(t,x,\nu)=x\cosh(t)-\mathrm{i}\nu t$ and i is the imaginary number. The first solution $K_{\mathrm{i}\nu}(x)$ stands out in the family of Bessel functions as a real valued solution to (5.1) whereas all the remaining solutions are complex valued. Motivated in part by the fact that $K_{\mathrm{i}\nu}(x)$ is the imaginary part of $I_{\mathrm{i}\nu}(x)$ up to a multiplicative factor, recent investigations have been carried out on the real part $I_{i\nu}(x)$, denoted by $L_{i\nu}(x)$, in [14,30]. We use the definition[4] given in [30]

$$L_{i\nu}(x) = \tfrac{1}{2}\{I_{i\nu}(x) + I_{-i\nu}(x)\}. \tag{5.3}$$

Any of $L_{i\nu}(x)$, $K_{i\nu}(x)$, and $I_{i\nu}(x)$ can be constructed from the remaining two functions by the identity

$$I_{i\nu}(x) = L_{i\nu}(x) - \mathrm{i}\,\frac{\sinh(\nu\pi)}{\pi}\,K_{i\nu}(x). \tag{5.4}$$

The function $L_{i\nu}(x)$ shares two important properties with $K_{i\nu}(x)$: they are both real valued solutions to Eq. (5.1), and both are even functions of ν. In fact, the two form a numerically satisfactory pair of solutions to (5.1) in the sense of Miller [23]: $L_{i\nu}(x)$ is exponentially large and $K_{i\nu}(x)$ is exponentially small in a neighborhood of $x=\infty$, and both are oscillatory in a neighborhood of $x=0$ with a phase difference of $\frac{1}{2}\pi$. It follows from (5.4) that the Wronskian of the system is $\mathscr{W}\{K_{i\nu}(x), L_{i\nu}(x)\} = \mathscr{W}\{K_{i\nu}(x), I_{i\nu}(x)\} = x^{-1}$.

We cannot apply Laplace's method as stated in this paper to the integrals in (5.2) directly to obtain asymptotic expansions for $K_{i\nu}(x), L_{i\nu}(x)$, and their derivatives since the exponent function $\phi(t,x,\nu)$ is complex valued. The usual approach to the problem is to use the *Method of Steepest Descents*, see [3, Chap. 6.6], [15, Chap. 2.5], or [24, Chap. 4.10]. This method deforms the contour of integration to paths in the complex plane along which the integrand is nonoscillatory, i.e., paths where $\mathscr{I}\{\phi(t,x,\nu)\}$ is constant. Once this is accomplished, then direct application of Laplace's method will yield the desired asymptotic expansion. The steepest descent paths for the functions $K_{i\nu}(x), L_{i\nu}(x)$ and their derivatives are thoroughly examined in [30] with the intention of computing these functions by numerical integration along these contours. This has been recently undertaken in part in [18]. We quote integral representations from [30]. Throughout the remainder of this section, the definitions of f, g, h, μ and ψ differ from those in the previous sections and closely follow the definitions in [30].

5.1. Debye expansions for $x > \nu$

It is convenient to write $\nu = x\sin\theta$, where θ is a fixed parameter such that $0 \leqslant \theta < \frac{1}{2}\pi$. For this parameter regime, the function $\phi(t,x,\nu)$ has pairs of second order saddle points[5] at $t_n = \mathrm{i}[(-)^n\theta + n\pi]$, where n is any integer, which lie on the imaginary t axis a distance $\frac{1}{2}\pi - \theta$ away from the points $\bar{t}_n = \mathrm{i}(\frac{1}{2}\pi + 2n\pi)$.

5.1.1. The functions $K_{i\nu}(\nu\csc\theta)$ and $K'_{i\nu}(\nu\csc\theta)$

It suffices to deform the integration contour so that it passes only through the saddle point at $t = t_0$. By writing $t = \tau + \mathrm{i}\sigma$, we parameterize the steepest descent contour through $t = t_0$ using τ by

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[4] We draw the reader's attention to the fact that $L_{i\nu}(x)$ as defined in [30] differs from that given in [14] by a multiplicative factor of $\pi/\sinh(\nu\pi)$.

[5] A function $f(t)$ has a "saddle point of the mth order" at $t = t_s$ if $f'(t_s) = f''(t_s) = \cdots = f^{(m-1)}(t_s) = 0$ and $f^{(m)}(t_s) \neq 0$.

defining

$$\sigma(\tau;\theta)=\arcsin\left(\sin\theta\,\frac{\tau}{\sinh\tau}\right),\qquad \psi(\tau;\theta)=\cosh\tau\cos(\sigma(\tau;\theta))+\sigma(\tau;\theta)\sin\theta. \tag{5.5}$$

Then the first of (5.2) becomes

$$K_{\mathrm{i}\nu}(\nu\csc\theta)=\int_0^\infty \mathrm{e}^{-\nu\csc\theta\,\psi(\tau;\theta)}\,\mathrm{d}\tau. \tag{5.6}$$

We implement Laplace's method using our *Maple* codes with $q(\tau)=1$, $p(\tau)=\psi(\tau;\theta)$ and $\tau=0$ to find that for fixed θ, the resulting expansions can be rearranged to obtain

$$K_{\mathrm{i}\nu}(\nu\csc\theta)\sim \mathrm{e}^{-\nu(\cot\theta+\theta)}\sqrt{\frac{\pi}{2\nu\cot\theta}}\sum_{s=0}^{\infty}(-)^s\,\frac{V_s(\tan\theta)}{\nu^s},\quad \nu\to+\infty, \tag{5.7}$$

where the polynomials $V_s(q)$ are given in [14] as

$$V_0(q)=1,\quad V_{s+1}(q)=\frac{1}{2}\,q^2(q^2+1)V_s'(q)+\frac{1}{8}\int_0^q V_s(t)(1+5t^2)\,\mathrm{d}t, \tag{5.8}$$

$s=0,1,2,\ldots$. This expansion was first given by [2] with different notation. It also appears in [21] whose notation differs from both ours and that in [2]. It was necessary during the computations to explicitly assume the conditions $0<\theta<\frac{1}{2}\pi$.

The integral representation for $K_{\mathrm{i}\nu}'(x)$ deformed to the steepest descent contour in this parameter regime is

$$K_{\mathrm{i}\nu}'(\nu\csc\theta)=\int_0^\infty g(\tau;\theta)\mathrm{e}^{-\nu\csc\theta\,\psi_d}\,\mathrm{d}\tau, \tag{5.9}$$

where

$$g(\tau;\theta)=\frac{\sin\theta\sin\sigma_d-\cosh\tau}{\cos\sigma_d},$$

$\sigma_d=\sigma(\tau;\theta)$ and $\psi_d=\psi(\tau;\theta)$. Our *Maple* code with $q(\tau)=g(\tau,\theta)$, $p(\tau)=\psi(\tau;\theta)$, and $\tau=0$ again yields, as $\nu\to+\infty$

$$K_{\mathrm{i}\nu}'(\nu\csc\theta)\sim -\mathrm{e}^{-\nu(\cot\theta+\theta)}\sin\theta\sqrt{\frac{\pi\cot\theta}{2\nu}}\sum_{s=0}^{\infty}(-)^s\,\frac{W_s(\tan\theta)}{\nu^s}, \tag{5.10}$$

where the polynomials $W_s(q)$ are

$$W_0(q)=1,\quad W_s(q)=V_s(q)-q(q^2+1)\{\tfrac{1}{2}V_{s-1}(q)+qV_{s-1}'(q)\}, \tag{5.11}$$

$s=1,2,3,\ldots$. Again, this first appeared in [2] with different notation. The notation for the polynomials $W_s(q)$ is new and is motivated by comparison with similar well-known expansions for the unmodified Bessel functions $J_\nu(\nu\csc\theta), Y_\nu(\nu\csc\theta)$, and their derivatives [1].

5.1.2. The functions $L_{i\nu}(\nu\csc\theta)$ and $L'_{i\nu}(\nu\csc\theta)$

The integral representation of $L_{i\nu}(x)$ is obtained that for $I_{i\nu}(x)$. The steepest descent contour for $I_{i\nu}(x)$ passes through the saddle points t_{-1}, t_0, and t_1. By deforming the second of (5.2) to this contour, separating real and imaginary parts, and using (5.4), [30] finds that

$$L_{i\nu}(\nu\csc\theta) = \frac{1}{2\pi}\int_{-\pi-\theta}^{\pi-\theta} e^{\nu\csc\theta(\cos\alpha+\alpha\sin\theta)}\,d\alpha - \frac{1}{2\pi}e^{\nu\pi}\int_0^\infty e^{-\nu\csc\theta\,\psi(\tau;\theta)}\frac{d\sigma}{d\tau}\,d\tau + \frac{1}{2\pi}e^{-\nu\pi}\int_0^\infty e^{-\nu\csc\theta\,\psi(\tau;\theta)}\frac{d\sigma}{d\tau}\,d\tau, \tag{5.12}$$

where

$$\frac{d\sigma}{d\tau} = \tan\sigma\left(\frac{1}{\tau} - \coth(\tau)\right).$$

We symbolically write the above expression as $L_{i\nu}(\nu\csc\theta) = L_1 - L_2 + L_3$. It is easy to see that as $\nu \to +\infty$,

$$L_1 = O(\exp\{\nu(\theta+\cot\theta)\}/\sqrt{\nu}),$$

$$L_2 = O(\exp\{\nu(\pi-\theta-\cot\theta)\}/\sqrt{\nu}),$$

$$L_3 = O(\exp\{-\nu(\pi+\theta+\cot\theta)\}/\sqrt{\nu}).$$

Obviously, L_3 is exponentially small for all values of θ and thus is negligible compared to L_1 and L_2. The integral L_2 is exponentially small for $0 \leqslant \theta < 0.3432\ldots$ and is exponentially large for $0.3432\ldots < \theta < \frac{1}{2}\pi$. In principle, $L_1 + L_2 \sim L_1$ for $\theta < \frac{1}{2}\pi$. However, L_2 is not negligible in the limit $\theta \to \frac{1}{2}\pi^-$. The main contribution to L_1 occurs at $\alpha = \theta$, and thus it needs to be rearranged so that we can use our *Maple* codes. The final form of the expansion is

$$L_{i\nu}(\nu\csc\theta) \sim e^{\nu(\theta+\cot\theta)}\frac{1}{\sqrt{2\nu\pi\cot\theta}}\sum_{s=0}^\infty \frac{V_s(\tan\theta)}{\nu^s} + e^{\nu(\pi-\theta-\cot\theta)}\frac{3\nu\cot^2\theta}{2\pi}\sum_{s=0}^\infty(-)^s\frac{\tilde V_s(\tan\theta)}{\nu^s} + O(e^{-\nu(\pi+\theta+\cot\theta)}) \quad \nu\to+\infty. \tag{5.13}$$

The polynomials $\tilde V_s(q)$ do not appear to follow a recognizable pattern. Therefore, we list the first few:

$$\tilde V_0(q) = 1,$$

$$\tilde V_1(q) = \tfrac{4}{5}q + \tfrac{8}{9}q^3,$$

$$\tilde V_2(q) = \tfrac{36}{35}q^2 + \tfrac{28}{9}q^4 + \tfrac{56}{27}q^6,$$

$$\tilde V_3(q) = \tfrac{64}{35}q^3 + \tfrac{368}{35}q^5 + \tfrac{448}{27}q^7 + \tfrac{640}{81}q^9.$$

An expansion similar to this was first obtained in [14] using a differential equation approach neglecting the contribution of L_2. That approach is more powerful in that the formulas for $V_j(q)$ in (5.8) are obtained directly from the differential equation. This form is not obtainable using the approach described here as is reflected in the lack of formulas for $\tilde{V}_j(q)$.

We compute an asymptotic expansion for $L'_{i\nu}(\nu\csc\theta)$ by using our *Maple* codes on the integral representation

$$L'_{i\nu}(\nu\csc\theta) = \frac{1}{2\pi}\int_{-\pi-\theta}^{\pi-\theta}\cos\alpha \mathrm{e}^{\nu\csc\theta(\cos\alpha+\sin\theta\alpha)}\,\mathrm{d}\alpha + \frac{\mathrm{e}^{\nu\pi}}{2\pi}\int_0^\infty \mathrm{e}^{-\nu\csc\theta\psi_d}h(\tau)\,\mathrm{d}\tau - \frac{\mathrm{e}^{-\nu\pi}}{2\pi}\int_0^\infty \mathrm{e}^{-\nu\csc\theta\psi_d}h(\tau)\,\mathrm{d}\tau, \tag{5.14}$$

where

$$h(\tau) = \sin\sigma_d\left(\frac{\cosh\tau}{\tau} - \frac{1}{\sinh\tau}\right).$$

As before, the third integral is exponentially small for all values of the parameter θ and the second integral is negligible for $\theta < \frac{1}{2}\pi$, though not in the limit as $\theta \to \frac{1}{2}\pi^-$. Using our *Maple* codes on these integrals, we find that

$$L'_{i\nu}(\nu\csc\theta) \sim \mathrm{e}^{\nu(\theta+\cot\theta)}\sqrt{\frac{\cot\theta}{2\nu\pi}}\sum_{s=0}^{\infty}\frac{W_s(\tan\theta)}{\nu^s} + \mathrm{e}^{\nu(\pi-\theta-\cot\theta)}\frac{\tan\theta\sin\theta}{3\pi}\sum_{s=0}^{\infty}(-)^s\frac{\tilde{W}_s(\tan\theta)}{\nu^s} + \mathrm{O}(\mathrm{e}^{-\nu(\pi+\theta+\cot\theta)}) \quad \nu \to +\infty. \tag{5.15}$$

Again, we list the first few polynomials $\tilde{W}_s(q)$:

$$\tilde{W}_0(q) = 1,$$

$$\tilde{W}_1(q) = \tfrac{3}{5}q + \tfrac{5}{9}q^3,$$

$$\tilde{W}_2(q) = \tfrac{24}{35}q^2 + \tfrac{28}{15}q^4 + \tfrac{32}{27}q^6,$$

$$\tilde{W}_3(q) = \tfrac{8}{7}q^3 + \tfrac{92}{15}q^5 + \tfrac{28}{3}q^7 + \tfrac{352}{81}q^9.$$

This expansion has not appeared in the existing literature to our knowledge.

5.2. Debye expansions for $x < \nu$

In this parameter regime, the real valued modified Bessel functions are oscillatory. By writing $\nu = x\cosh\mu$, where $\mu > 0$, we find that the pairs of second order saddle points now lie on both sides of the imaginary axis on the lines $\mathscr{I}(t) = \frac{1}{2}\pi + 2n\pi$ a distance μ away from the axis. For the sake of brevity, we state the results obtained by our *Maple* procedures and refer the reader to [30] for details of the integral representations.

5.2.1. The functions $K_{i\nu}(\nu\,\mathrm{sech}\,\mu)$ and $K'_{i\nu}(\nu\,\mathrm{sech}\,\mu)$

Let

$$\Pi_1 = \int_\mu^\infty \mathrm{e}^{-\nu\psi(\tau,f(\tau))}(1-f'(\tau))\,\mathrm{d}\tau + \int_{\pi/2}^{5\pi/2}\mathrm{e}^{-\nu\psi(g(\sigma),\sigma)}(1-g'(\sigma))\,\mathrm{d}\sigma,$$

$$\Pi_2 = \int_{\mu}^{\infty} \mathrm{e}^{-\nu\psi(\tau,f(\tau))}(1+f'(\tau))\,\mathrm{d}\tau - \int_{\pi/2}^{5\pi/2} \mathrm{e}^{-\nu\psi(g(\sigma),\sigma)}(1+g'(\sigma))\,\mathrm{d}\sigma,$$

$$\Pi_3 = \int_{\mu}^{\infty} \mathrm{e}^{-\nu\psi(\tau,f(\tau))}(A(\tau)-C(\tau))\,\mathrm{d}\tau - \int_{\pi/2}^{5\pi/2} \mathrm{e}^{-\nu\psi(g(\sigma),\sigma)}(\bar{B}(\sigma)-\bar{D}(\sigma))\,\mathrm{d}\sigma,$$

$$\Pi_4 = \int_{\mu}^{\infty} \mathrm{e}^{-\nu\psi(\tau,f(\tau))}(A(\tau)+C(\tau))\,\mathrm{d}\tau - \int_{\pi/2}^{5\pi/2} \mathrm{e}^{-\nu\psi(g(\sigma),\sigma)}(\bar{B}(\sigma)+\bar{D}(\sigma))\,\mathrm{d}\sigma,$$

where

$$\psi(\tau,\sigma) = \operatorname{sech}\mu\cosh\tau\cos\sigma + \sigma - \tfrac{1}{2}\pi,$$

$$\sin(f(\tau)) = \frac{(\tau-\mu)\cosh\mu + \sinh(\mu)}{\sinh\tau},$$

$$g(\sigma) = f^{-1}(\sigma),$$

$$A(\tau) = -\cosh\tau\cos(f(\tau)) + \sinh(\tau)\sin(f(\tau))f'(\tau),$$

$$C(\tau) = -\sinh\tau\sin(f(\tau)) - \cosh(\tau)\cos(f(\tau))f'(\tau),$$

$$\bar{B}(\sigma) = A(g(\sigma))g'(\sigma),$$

$$\bar{D}(\sigma) = C(g(\sigma))g'(\sigma).$$

Then, if $\chi = \nu(\tanh\mu - \mu)$, we rewrite the integrals in [30] in so that they coincide with the notation of [14] as

$$K_{\mathrm{i}\nu}(\nu\operatorname{sech}\mu) = \frac{\mathrm{e}^{-\nu\pi/2}}{\sqrt{2}}\left\{\cos\left(-\chi-\frac{1}{4}\pi\right)\Pi_1 - \sin\left(-\chi-\frac{1}{4}\pi\right)\Pi_2\right\} + \mathrm{O}(\mathrm{e}^{-5\nu\pi/2}), \tag{5.16}$$

$$K'_{\mathrm{i}\nu}(\nu\operatorname{sech}\mu) = \frac{\mathrm{e}^{-\nu\pi/2}}{\sqrt{2}}\left\{\cos\left(-\chi-\frac{1}{4}\pi\right)\Pi_3 - \sin\left(-\chi-\frac{1}{4}\pi\right)\Pi_4\right\} + \mathrm{O}(\mathrm{e}^{-5\nu\pi/2}). \tag{5.17}$$

A surface plot of $\psi(\tau,\sigma)$ with the curve $\psi(\tau,f(\tau))$ superimposed shows that ψ has an absolute minimum at $\tau=\mu$, $\sigma=\frac{1}{2}\pi$ on the integration intervals. Note that $g(\sigma)$ does not have closed form solution. Therefore, we obtain its series expansion about $\frac{1}{2}\pi$ by reversion of the expansion of $f(\tau)$ about μ. Rearrangement of the expansions returned by our Laplace's method codes yields, for μ fixed as $\nu\to+\infty$,

$$\Pi_1 \sim \frac{2\sqrt{\pi\coth\mu}}{\sqrt{\nu}}\sum_{s=0}^{\infty}\frac{V_{2s}(\mathrm{i}\coth\mu)}{\nu^{2s}}, \tag{5.18}$$

$$\Pi_2 \sim -\frac{2\sqrt{\pi\coth\mu}}{\sqrt{\nu}}\sum_{s=0}^{\infty}\frac{\mathrm{i}V_{2s+1}(\mathrm{i}\coth\mu)}{\nu^{2s+1}}, \tag{5.19}$$

$$\Pi_3 \sim -\frac{2\sqrt{\pi}\cosh\mu}{\sqrt{v\coth\mu}} \sum_{s=0}^{\infty} \frac{\mathrm{i}W_{2s+1}(\mathrm{i}\coth\mu)}{v^{2s+1}}, \tag{5.20}$$

$$\Pi_4 \sim -\frac{2\sqrt{\pi}\cosh\mu}{\sqrt{v\coth\mu}} \sum_{s=0}^{\infty} \frac{W_{2s}(\mathrm{i}\coth\mu)}{v^{2s}}. \tag{5.21}$$

5.2.2. The functions $L_{\mathrm{i}v}(v\,\mathrm{sech}\,\mu)$ and $L'_{\mathrm{i}v}(v\,\mathrm{sech}\,\mu)$

Using the notation of the previous subsection, it can be shown that

$$L_{\mathrm{i}v}(v\,\mathrm{sech}\,\mu) = \frac{\mathrm{e}^{v\pi/2}}{2\sqrt{2\pi}} \left\{ -\cos\left(-\chi - \frac{1}{4}\pi\right)\Pi_2 - \sin\left(-\chi - \frac{1}{4}\pi\right)\Pi_1 \right\} + \mathrm{O}(\mathrm{e}^{-v\pi/2}), \tag{5.22}$$

$$L'_{\mathrm{i}v}(v\,\mathrm{sech}\,\mu) = \frac{\mathrm{e}^{v\pi/2}}{2\sqrt{2\pi}} \left\{ \cos\left(-\chi - \frac{1}{4}\pi\right)\Pi_3 + \sin\left(-\chi - \frac{1}{4}\pi\right)\Pi_4 \right\} + \mathrm{O}(\mathrm{e}^{-v\pi/2}). \tag{5.23}$$

Again, the expansion for $L'_{\mathrm{i}v}(v\,\mathrm{sech}\,\mu)$ is new.

5.3. Expansions at the turning point $x = v$

The turning point is unique in that the pairs second order saddle points coalesce into third order saddle points at $\bar{t}_n = \mathrm{i}(\frac{1}{2}\pi + 2n\pi)$, where n is an integer. The expansions presented below are new, though those for $K_{\mathrm{i}v}(v)$ and $K'_{\mathrm{i}v}(v)$ can be deduced from those in [2].

5.3.1. The functions $K_{\mathrm{i}v}(v)$ and $K'_{\mathrm{i}v}(v)$

Careful analysis of the steepest descent contours for this case yields the integrals in (5.6) and (5.9) with θ replaced by $\frac{1}{2}\pi$. (Note that this holds only for the integral representations and not for the Debye expansions.) Therefore, we simply state the obtained expansions

$$K_{\mathrm{i}v}(v) = \pi\mathrm{e}^{-\pi v/2} \left\{ \frac{2^{1/3}\mathrm{Ai}(0)}{v^{1/3}} S(\mathrm{i}v) + \frac{2^{2/3}\mathrm{Ai}'(0)}{v^{5/3}} T(\mathrm{i}v) \right\} \tag{5.24}$$

in the notation of [31], where the functions $S(\mu)$ and $T(\mu)$ have the expansions as $\mu \to +\infty$

$$S(\mu) \sim 1 - \frac{1}{225}\frac{1}{\mu^2} + \frac{151439}{218295000}\frac{1}{\mu^4} - \frac{887278009}{2504935125000}\frac{1}{\mu^6} + \mathrm{O}\left(\frac{1}{\mu^8}\right),$$

$$T(\mu) \sim \frac{1}{70} - \frac{1213}{1023750}\frac{1}{\mu^2} + \frac{16542537833}{3774320550000}\frac{1}{\mu^4} - \frac{9597171184603}{25476663712500000}\frac{1}{\mu^6} + \mathrm{O}\left(\frac{1}{\mu^8}\right),$$

and where $\mathrm{Ai}(0)$ and $\mathrm{Ai}'(0)$ are the values of the Airy function $\mathrm{Ai}(x)$ and its derivative at the origin, respectively.

$$K'_{\mathrm{i}v}(v) = \pi\mathrm{e}^{-\pi v/2} \left\{ \frac{2^{2/3}\mathrm{Ai}'(0)}{v^{2/3}} U(\mathrm{i}v) - \frac{2^{1/3}\mathrm{Ai}(0)}{v^{4/3}} V(\mathrm{i}v) \right\}, \tag{5.25}$$

where we use the notation of [31]

$$U(\mu) \sim 1 + \frac{23}{3150}\frac{1}{\mu^2} - \frac{604523}{644962500}\frac{1}{\mu^4} + \frac{2264850139339}{5095332742500000}\frac{1}{\mu^6} + \mathrm{O}\left(\frac{1}{\mu^8}\right),$$

$$V(\mu) \sim \frac{1}{5} - \frac{947}{346500}\frac{1}{\mu^2} + \frac{11192989}{18555075000}\frac{1}{\mu^4} - \frac{100443412440047}{262141460831250000}\frac{1}{\mu^6} + \mathrm{O}\left(\frac{1}{\mu^8}\right).$$

5.3.2. The functions $L_{\mathrm{i}\nu}(\nu)$ and $L'_{\mathrm{i}\nu}(\nu)$

Again, the integrals here are again those given in (5.12) and (5.14) with θ replaced by $\frac{1}{2}\pi$. The contribution of the second term in the integral representations of $L_{\mathrm{i}\nu}(x)$ and $L'_{\mathrm{i}\nu}(x)$ is now critical.

$$L_{\mathrm{i}\nu}(\nu) = \frac{1}{2}\,\mathrm{e}^{\nu\pi/2}\left\{\frac{2^{1/3}\mathrm{Bi}(0)}{\nu^{1/3}}\,S(\mathrm{i}\nu) + \frac{2^{2/3}\mathrm{Bi}'(0)}{\nu^{5/3}}\,T(\mathrm{i}\nu)\right\} + \mathrm{O}(\mathrm{e}^{-3\nu\pi/2}), \tag{5.26}$$

where $\mathrm{Bi}(0)$ and $\mathrm{Bi}'(0)$ are the values of the Airy function $\mathrm{Bi}(x)$ and its derivative at the origin, respectively. Finally,

$$L'_{\mathrm{i}\nu}(\nu) = \frac{1}{2}\,\mathrm{e}^{\nu\pi/2}\left\{\frac{2^{2/3}\mathrm{Bi}'(0)}{\nu^{2/3}}\,U(\mathrm{i}\nu) - \frac{2^{1/3}\mathrm{Bi}(0)}{\nu^{4/3}}\,V(\mathrm{i}\nu)\right\} + \mathrm{O}(\mathrm{e}^{-3\nu\pi/2}). \tag{5.27}$$

5.4. Discussion

It comes as no surprise that the asymptotic expansions obtained in this section are similar to those for the unmodified Bessel functions $J_\nu(x), Y_\nu(x)$ and their derivatives. Compare the expansions obtained here in [2,14] with those in [1, Chap. 9.3]. This similarity is due to the fact that the modified and unmodified Bessel functions are related to the Hankel functions $H_\mu^{(j)}(\zeta)$ of the first $(j=1)$ and second $(j=2)$ kinds:

$$J_\mu(\zeta) = \tfrac{1}{2}\{H_\mu^{(1)}(\zeta) + H_\mu^{(2)}(\zeta)\}, \quad Y_\mu(\zeta) = \frac{1}{2i}\{H_\mu^{(1)}(\zeta) - H_\mu^{(2)}(\zeta)\},$$

$$K_\mu(\zeta) = \tfrac{1}{2}\pi\mathrm{i}\mathrm{e}^{\mu\pi\mathrm{i}/2}H_\mu^{(1)}(\mathrm{i}\zeta), \quad L_\mu(\zeta) = \tfrac{1}{2}\{\mathrm{e}^{\mu\pi\mathrm{i}/2}\cosh(\mu\pi\mathrm{i})H_\mu^{(1)}(\mathrm{i}\zeta) + \mathrm{e}^{-\mu\pi\mathrm{i}/2}H_\mu^{(2)}(\mathrm{i}\zeta)\},$$

where μ and ζ are dummy variables.

It must be emphasized that the *Maple* routines do not return clean and neat expansions as those given here. On the contrary, the returned expansions are quite difficult to read. A significant amount of energy was spent transforming the returned expansions into the forms presented here. Furthermore, we cannot conclude from the work presented here that the stated forms for the asymptotic expansions are valid for infinitely many terms. The computations which obtained our results quickly overwhelmed our computational resources. For example, reversion of $f(\tau)$ in Section 5.2 to $\mathrm{O}((\sigma - \frac{1}{2}\pi)^{12})$ by itself lasted several hours on a 500 MHz *Compaq* Alpha workstation with 640 MB of RAM, whereas the simplification of each coefficient required twice as much time on a *Compaq* Alpha server with 3.13 GB of available RAM and a dedicated processor. We deduce the forms of the expansions by careful examination of the first few terms which we could compute using our codes and by comparison with the corresponding expansions of the unmodified Bessel functions. This is a drawback of the approach based on integral representations to deriving asymptotic expansions.

Finally, we note that (possibly complex-valued) asymptotic expansions may be obtained directly from (5.2) using saddle point methods. See Section 2.6 of [24]. However, in order to obtain the expansions in the form presented here, and in particular the exponentially large but often negligible contributions for $L_{i\nu}(\nu \csc\theta)$ and $L'_{i\nu}(\nu \csc\theta)$, analysis on the steepest descent paths is required.

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References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, Mathematical Tables, US Government Printing Office, Washington DC, 10th printing Edition, 1972.
[2] C.B. Balough, Asymptotic expansions of the modified Bessel function of the third kind of imaginary order, SIAM J. Appl. Math. 15 (1967) 1315–1323.
[3] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, Inc., New York, 1978, Reprinted by Springer, New York, 2000.
[4] F. Bergeron, S. Plouffe, Computing the generating function of a series given its first few terms, Experiment. Math. 1 (1992) 307–312.
[5] W.G. Bickley, J.C.P. Miller, Notes on the reversion of a series, Philos. Mag. 34 (1949) 35–36.
[6] G.R. Blakley, Formal solution of nonlinear simultaneous equations: reversion of series in several variables, Duke Math. J. 31 (1964) 347–357.
[7] R.P. Brent, H.T. Kung, Fast algorithms for composition and reversion of multivariate power series, in: Proceedings of the Conference on Theoretical Computer Science, University Waterloo, University of Waterloo, Waterloo, 1977, pp. 149–158.
[8] R.P. Brent, H.T. Kung, Fast algorithms for manipulating formal power series, J. ACM 25 (1978) 581–595.
[9] F. Chang, Power series unification and reversion, Appl. Math. Comput. 23 (1987) 7–23.
[10] B.W. Char, K.O. Geddes, G.H. Gonnet, B. Leong, M. Monaghan, S. Watt, MAPLE V Library Reference Manual, Springer, New York, 1991.
[11] C.C. Cheng, J.H. McKay, J. Towber, S.S. Wang, D.L. Wright, Reversion of power series and the extended Raney coefficients, Trans. Amer. Math. Soc. 349 (1997) 1769–1782.
[12] H. Chernoff, A note on the inversion of power series, Math. Comput. 2 (1947) 331–335.
[13] E.T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford University Press, London, 1935.
[14] T.M. Dunster, Bessel functions of purely imaginary order, with an application to second-order linear differential equations having a large parameter, SIAM J. Math. Anal. 21 (1990) 995–1018.
[15] A. Erdélyi, Asymptotic Expansions, Dover Publications, New York, 1956.
[16] B.R. Fabijonas, F.W.J. Olver, On the reversion of an asymptotic expansion and the zeros of the Airy functions, SIAM Rev. 41 (1999) 762–773.

[6] Certain commercial equipment, instruments, or materials are identified in this paper to foster understanding. Such identification does not imply recommendation or endorsement by the NIST, nor does it imply that the materials or equipment identified are necessarily the best available for the purpose.

[17] D.C. Fielder, Tabulation of coefficients for operations on Taylor series, Math. Comput. 14 (1960) 339–345.
[18] A. Gil, J. Segura, N.M. Temme, Evaluation of the modified Bessel function of the third kind of imaginary orders, J. Comput. Phys. (2002), to appear.
[19] A.J. Goldstein, A.D. Hall, Solutions to a problem in power series reversion, SIAM J. Math. Anal. 6 (1975) 192–198.
[20] J.L. Lagrange, Nouvelle Méthode pour Résoudre les Équations Littérales par le moyen des séries, Mémoires de l'Académy des Sciences et Belles-Lettres de Berlin 24 (1770), As found in Oeuvres de Lagrange publiées par le soins de M.J.-A. Serret, Vol. III, Gauthier-Villars, Paris, 1869, pp. 4–73; reprinted by Georg Olms Verlag, Hildesheim, 1973.
[21] W. Magnus, F. Oberherttinger, R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer, Berlin, 1966.
[22] J. McMahon, On the general term in the reversion of series, Bull. Amer. Math. Soc. 3 (1894) 170–172.
[23] J.C.P. Miller, On the choice of standard solutions for a homogeneous linear differential equation of the second order, Quart. J. Mech. Appl. Math. 3 (1950) 225–235.
[24] F.W.J. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974, Reprinted by AK Peters, Wellesley, 1997.
[25] J.K. Percus, A note on extension of the Lagrange inversion formula, Commun. Pure Appl. Math. 17 (1964) 137–146.
[26] G.N. Raney, Functional composition patterns and power series reversion, Trans. ACM 94 (1960) 441–451.
[27] B. Salvy, Fast computation of some asymptotic functional inverses, J. Symbolic Comput. 17 (1994) 227–236.
[28] B. Salvy, J. Shackell, Symbolic asymptotics: multiseries of inverse functions, J. Symbolic Comput. 27 (1999) 543–563.
[29] B. Salvy, P. Zimmermann, GFUN: a Maple package for the manipulation of generating and holonomic functions in one variable, ACM Trans. Math. Software 20 (1994) 163–177.
[30] N.M. Temme, Steepest descent paths for integrals defining the modified Bessel functions of imaginary order, Methods Appl. Anal. 1 (1994) 14–24.
[31] N.M. Temme, Numerical algorithms for uniform Airy-type asymptotic expansions, Numer. Algorithms 15 (1997) 207–225.
[32] H.C. Thacher, Algorithm 273, SERREV, Commun. ACM 9 (1966) 11.
[33] H.C. Thacher, Solutions of transcendental equations by series reversion, Commun. ACM 9 (1966) 10–11.
[34] J.A. Tyrrell, Reversion of a formal power series, Mathematika 9 (1962) 88–94.
[35] C.E. Van Orstrand, Reversion of power series, Philos. Mag. 19 (1910) 366–376.
[36] G.N. Watson, The harmonic functions associated with the parabolic cylinder, Proc. London Math. Soc. 17 (1918) 116–148.
[37] E.T. Whittaker, On the reversion of series, Gaz. Math. 50 (1951) 1.